ECON 186 Class Notes: Probability and Statistics Part 1

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Integration

- Let us return to the other side of calculus, integration, for several reasons.

  ▸ First, integrals are often used in dynamic analysis. In the static problems we have considered so far, we assume that equilibrium is just achieved, but dynamic analysis looks at how variables move in each time period on their way to equilibrium.

  ▸ Second, and most important for our present use, is that integrals are used to denote the probability of a continuous statistical distribution due to their interpretation as the area under a curve.

  ▸ Third, integrals are used to denote an infinite continuum of individuals or other economic units in the specification of various models.

Integration

- Informally, a definite integral is the signed area of the region in the xy-plane bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$, such that area above the x-axis adds to the total, and that below the x-axis subtracts from the total.
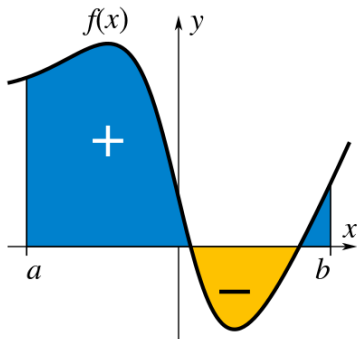
Integration

- The definite integral thus described is specified as

$$\int_a^b f(x)dx$$

- where $f(x)$ is the integrand, x is the variable of integration, a is the lower limit of integration, and b is the upper limit of integration.

- An antiderivative is a function $F(x)$ whose derivative is $f(x)$. An indefinite integral is the set of all antiderivatives of $f(x)$, so

$$\frac{d}{dx}F(x) = f(x) \rightarrow F(x) + c = \int f(x)dx$$

Rules of Integration

- Rule I: The Power Rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

- Examples

$$\int x^3 dx = \frac{1}{4}x^4 + c$$

$$\int \sqrt{x^3} dx = \int x^{\frac{3}{2}} dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{5}\sqrt{x^5} + c$$

$$\int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + c = -\frac{1}{3x^3} + c$$

Rules of Integration

- Rule II: The Exponential Rule

$$\int f^{'}(x)e^{f(x)}dx = e^{f(x)} + c$$

- Example: Let $f(x) = x^2$, so $f^{'}(x) = 2x$. Then,

$$\int 2xe^{x^2}dx = e^{x^2} + c$$

- Rule III: The integral of a sum

$$\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$$

Examples of Integration

- Example:

$$\int \left(x^3 + x + 1\right) dx = \left(\frac{x^4}{4} + c_1\right) + \left(\frac{x^2}{2} + c_2\right) + (x + c_3)$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + x + c$$

Rules of Integration

- Rule IV: The integral of a multiple

$$\int kf(x)dx = k \int f(x)dx$$

- Rule V: The substitution rule

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$$

U-Substitution Example

- Example: Find $\int 6x^2 \left(x^3 + 2\right)^{99} dx$.

$$\int 6x^2 \left(x^3 + 2\right)^{99} dx$$

- Let $u = x^3 + 2$, then $\frac{du}{dx} = 3x^2 \rightarrow dx = \frac{du}{3x^2}$

$$\int 6x^2 u^{99} \frac{du}{3x^2} = \int 2u^{99} du = \frac{2}{100} u^{100} + c = \frac{1}{50} u^{100} + c$$

$$= \frac{1}{50} \left(x^3 + 2\right)^{100} + c$$

Rules of Integration

- Rule VI: The Logarithmic Rule

$$\int \frac{f'(x)}{f(x)} dx = ln\,|f(x)| + c \qquad for\ n = -1$$

- We can also restate the rule using u-substitution which makes the problem much easier to solve:

  - Let $u = f(x)$ be some function of x, then $\frac{du}{dx} = f'(x)$ and $du = f'(x)dx$. Then,

  $$\int \frac{1}{u} du = ln|u| + c$$

Examples of the Logarithmic Rule

- Examples:

$$\int \left(2e^{2x} + \frac{14x}{7x^2 + 5} \right) = \int 2e^{2x} dx + \int \frac{14x}{7x^2 + 5} dx$$

$$= e^{2x} + c_1 + ln\left(7x^2 + 5\right) + c_2$$

- Suppose that $f(x) = \frac{1}{4x-1}$. Then, $u = g(x) = 4x - 1$, so
$\frac{du}{dx} = 4 \rightarrow dx = \frac{1}{4} du$

$$\int \frac{1}{4x-1} dx = \int \frac{1}{u} \frac{1}{4} du = \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} ln|u| + c = \frac{1}{4} ln\left| \frac{1}{4x-1} \right| + c$$

Definite Integrals

- By the first fundemental theorem of calculus,

$$\int_b^a f(x)dx = F(b) - F(a)$$

- Example: Evaluate $\int_1^2 \left(2x^3 - 1\right)^2 \left(6x^2\right) dx$.

  ► The first step is to set $u = 2x^3 - 1$, then $\frac{du}{dx} = 6x^2 \rightarrow dx = \frac{du}{6x^2}$. So,

$$\int_1^2 \left(2x^3 - 1\right)^2 \left(6x^2\right) dx = \int_*^* u^2 du = \left.\frac{u^3}{3}\right|_*^* = \left.\frac{\left(2x^3 - 1\right)^3}{3}\right|_1^2$$

$$= \frac{1}{3}\left(2(8) - 1\right)^3 - \frac{1}{3}\left(2(1) - 1\right)^3 = \frac{1}{3}\left(15^3 - 1^3\right) = \frac{3374}{3}$$

Improper Integrals

- Thus far we have considered integrals with two finite limits of integration. However, suppose we have integrals of the form

$$\int_a^\infty f(x)dx \qquad and \qquad \int_{-\infty}^b f(x)dx$$

- If one or both limits are infinite, these are called improper integrals. In this case, the fundamental theorem of calculus tells us that

$$F(\infty) - F(a) \qquad and \qquad F(b) - F(-\infty)$$

- However, infinity is not a number, so we cannot evaluate $F(\infty)$. However, we can equivalently define the integrals above using limits as

$$\int_a^\infty f(x)dx \equiv \lim_{b \to \infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^b f(x)dx \equiv \lim_{a \to -\infty} \int_a^b f(x)dx$$

Improper Integrals

- If these limits exist, then the improper integrals are convergent, and the limit will yield the value of the integral.
- Furthermore, if both limits of integration are infinite, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

- It is important to note that c is simply an arbitrary constant, it does not matter what the value of c is. $c = 0$ is often used for convenience.
- Additionally, for the above improper integral to exist, it is necessary that both

$$\int_{-\infty}^{c} f(x)dx \qquad and \qquad \int_{c}^{\infty} f(x)dx$$

- are convergent.

Improper Integrals Examples

- Evaluate $\int_1^\infty \frac{dx}{x^2}$.
- First, we find that

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{b}$$

- Then, we can just take the limit of this expression

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1$$

Improper Integrals Examples

- Evaluate $\int_1^\infty \frac{dx}{x}$
- First we find

$$\int_1^\infty \frac{dx}{x} = \ln x \big|_1^b = \ln b - \ln 1 = \ln b$$

- If we take the limit as $b \to \infty$, it is clear that $\ln b \to \infty$, so the improper integral diverges, which means that the area under the curve of the function $y = \frac{1}{x}$ is infinite.

Improper Integrals Examples

- Evaluate $\int_{-\infty}^{\infty} x e^{-x^2} dx$.
- For convenience, let $c = 0$, then

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$$

- First, evaluate the first integral

$$\int_{-\infty}^{0} x e^{-x^2} dx = \lim_{a \to -\infty} \int_{a}^{0} x e^{-x^2} dx = \lim_{a \to -\infty} \left. -\frac{1}{2} e^{-x^2} \right|_{a}^{0}$$

$$= \lim_{a \to -\infty} -\frac{1}{2} \left(e^0 - e^{-a^2} \right) = \lim_{a \to -\infty} -\frac{1}{2} + \lim_{a \to -\infty} e^{-a^2} = -\frac{1}{2}$$
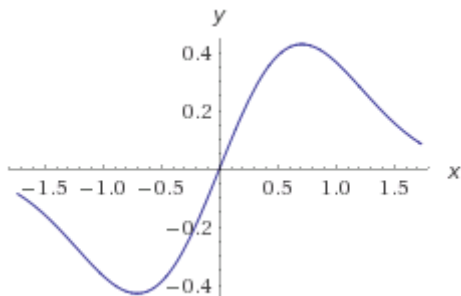
Improper Integrals Examples

- Next, evaluate the second integral

$$\int_0^\infty xe^{-x^2}dx = \lim_{b \to \infty} \int_0^b xe^{-x^2}dx = \lim_{b \to \infty} \left. -\frac{1}{2}e^{-x^2}\right|_0^b$$

$$= \lim_{b \to \infty} -\frac{1}{2}\left(e^{-b^2} - e^0\right) = \frac{1}{2}$$

- So, both integrals are convergent, which means the improper integral with two infinite limits of integration is convergent, and its value is

$$\int_{-\infty}^\infty xe^{-x^2}dx = \int_{-\infty}^0 xe^{-x^2}dx + \int_0^\infty xe^{-x^2}dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Improper Integrals Examples

Introduction to Probability Theory

- As economists, we would like to know the probability of economic events through repeated experimentation.

- The primary use of probability to economists is the application to statistics. Specifically, in econometrics, we would like to know that if we are given some sample, what is the probability that the result we are obtaining is actually the true result for the entire population.

- The outcome of an experiment is the result of the experiment, and it cannot be predicted with certainty.

- The sample space of an experiment is the collection of all possible outcomes.

- A random experiment is an experiment where the experimental units are assigned randomly across groups.

- Example: One example of a random experiment is the tossing of a coin.

  ▸ Sample space - Heads (H) and Tails (T).

Introduction to Probability Theory

- A Random Variable, which we denote with capital letters, is a function that assigns each outcome to one and only one real number.
- In the flipping a coin example, suppose there is a random variable X. Then, we can assign the random variable X values as follows:
    - $X = 0$ if T or $X = 1$ if H.
- Then, we can define the probability of a random variable:

$$Pr(X = x)$$

- is the probability that random variable X is equal to x, where x is a value of the random variable.
- In the flipping a coin example, $Pr(X = 0) = Pr(X = 1) = 0.5$ assuming a fair coin.

Introduction to Probability Theory

- There are two types of random variables, discrete and continuous.
  - Discrete: Set of outcomes is either finite or countably infinite.
    - Examples include the toss of a die, flip of a coin, the set of all integers.
  - Continuous: Set of outcomes is infinitely indivisible.
    - Examples include height, weight, time, the interval $(0,1)$. Note that we record height, weight, and time in certain discrete units because it makes them easier to understand for us, but they actually are continuous.

PMF's and PDF's

- $Pr(X = x)$ is called the probability mass function for discrete random variables and probability density function for continuous random variables. For convenience, let us define each specifically with the following notation:

- For a discrete random variable the probability mass function (pmf) is

$$p(x) = Pr(X = x) \qquad \text{where } x \in \mathbb{R}$$

- If $f(x)$ is the probability density function (pdf) of X, then X is a continuous random variable, and if X only takes values on the closed interval $[a, b] \in B$, then

$$Pr(X \in B) = Pr(a \leq X \leq b) = \int_a^b f(x) dx$$

$$Pr(X = a) = \int_a^a f(x) dx = 0$$

- This second condition means that the probability of any single point on a continuous distribution since the distribution can take on an uncountably infinite number of values.

Axioms of Probability

- Axioms of probability for discrete random variables:

$$1) \ 0 \leq Pr(X = x) \leq 1, \quad \forall x \in X$$

$$2) \sum_{x \in X} p(x) = 1$$

- Axioms of probability for continuous random variables:

$$1) \ f(x) \geq 0, \quad \forall x \in X$$

$$2) \int_{x \in X} f(x) dx = 1$$

- A useful property of continuous distributions is that
$Pr(a \leq X \leq b) = Pr(a < X \leq b) = Pr(a \leq X < b) = Pr(a < X < b)$
since $Pr(X = a) = 0$ for any a.

Examples

- Suppose that a random variable X has a discrete distribution with the pmf

$$f(x) = \begin{cases} cx & \text{for } x = 1, ..., 5 \\ 0 & \text{otherwise} \end{cases}$$

- Determine the value of the constant c.
  - By axiom 2 for discrete random variables:

$$\sum_{x=1}^{5} f(x) = c + 2c + 3c + 4c + 5c = 15c = 1 \rightarrow c = \frac{1}{15}$$

- Let X be a random variable with the pdf

$$f(x) = \begin{cases} \frac{2}{3} x^{-\frac{1}{3}} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Compute $Pr\left(0 \le X \le \frac{8}{27}\right)$.

$$Pr\left(0 \le X \le \frac{8}{27}\right) = \int_0^{\frac{8}{27}} \frac{2}{3} x^{-\frac{1}{3}} dx = x^{\frac{2}{3}} \Big]_0^{\frac{8}{27}} = \frac{4}{9}$$

Cumulative Distribution Function

- For discrete random variables the cumulative distribution function (CDF) is
$$F(x) = Pr(X \leq a) = \sum_{X \leq a} p(x)$$

- For continuous random variables the CDF is
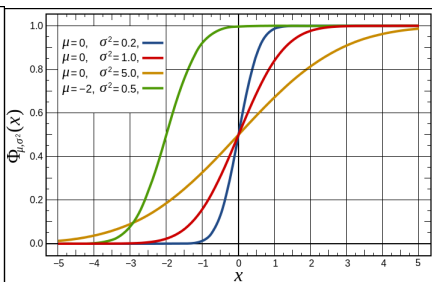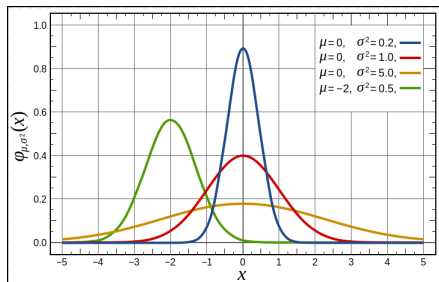$$F(x) = Pr(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

$$f(x) = \frac{dF(x)}{dx}$$

- So, the CDF is defined in either case as $Pr(X \leq a)$, which means that it is the total probability of X taking on all values less than or equal to some value a.

- Properties:
$$Pr(X > x) = 1 - F(x)$$
$$Pr(x_1 < X \leq x_2) = F(x_2) - F(x_1)$$

Examples of pdfs and cdfs

Examples

- Suppose that we consider the roll of a six-sided die. In this case,

$$p(x) = \frac{1}{6}, \quad \text{for } x = 1, 2, 3, 4, 5, 6$$

- What is the CDF $F(x)$?

$$F(2) = \sum_{X \leq 2} f(x) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$F(4) = \sum_{X \leq 4} f(x) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6}$$

$$F(x) = \frac{1}{6}x, \quad \text{for } x = 1, 2, 3, 4, 5, 6$$

Examples

- Suppose that the cdf of a random variable X is as follows:

$$\begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{9}x^2 & \text{for } 0 < x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

- Find the pdf of X.

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} 0 & \text{for } x < 0 \\ \frac{2}{9}x & \text{for } 0 < x < 3 \\ 0 & \text{for } x > 3 \end{cases}$$