Homework Day 8 Solutions - ECON 186

Problem 1. Chiang and Wainwright 11.4 #1, 3

1) First-order conditions:
$$f_1 = 2x_1 - 3x_2 = 0$$
$$f_2 = -3x_1 + 6x_2 + 4x_3 = 0$$
$$f_3 = 4x_2 + 12x_3 = 0$$

Since the three equations are independent, it is straightforward to show that the only solution is
$$x_1^* = x_2^* = x_3^* = 0$$

so that $z^* = 0$. Then, the Hessian is
$$\begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$$
$$|H_1| = 2 > 0, |H_2| = 3 > 0, |H_3| = 4 > 0$$

So, d^2z is positive definite and therefore $z^* = 0$ is a minimum.

3) First-order conditions:
$$f_1 = 2x_1 + x_3 = 0$$
$$f_2 = 2x_2 + x_3 = 1$$
$$f_3 = x_1 + x_2 + 6x_3 = 0$$

One way to find the solutions is to put the system of equations in a matrix and use Gaussian elimination.

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -11 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & 0 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & -2 & -11 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 11/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & -10 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 11/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -1/10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/20 \\ 0 & 1 & 0 & 11/20 \\ 0 & 0 & 1 & -1/10 \end{array}\right]$$

So $x_1^* = \frac{1}{20}$, $x_2^* = \frac{11}{20}$, $x_3^* = -\frac{1}{10}$

The Hessian is

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}$$

where

$$|H_1| = 2 > 0, |H_2| = 4 > 0, |H_3| = 20 > 0$$

so d^2z is positive definite and therefore $z^* = -\frac{11}{40}$ is a minimum.

Problem 2. Chiang and Wainwright 12.2 #1(d), 6

1)

d)

$$L = 7 - y + x^2 - \lambda(x + y)$$

First-order conditions:

$$L_x = 2x - \lambda = 0 \to \lambda = 2x$$

$$L_y = -1 - \lambda = 0 \to \lambda = -1$$

$$L_\lambda = -x - y = 0 \to x = -y$$

Combining the first two FOC's, $2x = -1 \to x^* = -\frac{1}{2}$. Then, combining with the last FOC: $y^* = -\left(-\frac{1}{2}\right) = \frac{1}{2}$. So, $z^* = 7 - \frac{1}{2} + \left(-\frac{1}{2}\right)^2 = \frac{28}{4} - \frac{2}{4} + \frac{1}{4} = \frac{27}{4}$

6) No, the sign of λ^* will be changed. The new λ^* is the negative of the old λ^*. Note however that Lagrangians are equivalently and often written as $L = f(x, y) - \lambda[c - g(x, y)]$.

Problem 3. Chiang and Wainwright 12.5 #1 (a,b)

1)

a)

$$L = (x + 2)(y + 1) + \lambda(130 - 4x - 6y)$$

b)

$$L_x = y + 1 - 4\lambda = 0 \to \lambda = \frac{y + 1}{4}$$

$$L_y = x + 2 - 6\lambda = 0 \to \lambda = \frac{x + 2}{6}$$

$$L_\lambda = 130 - 4x - 6y = 0$$

$$\frac{y + 1}{4} = \frac{x + 2}{6} \to \frac{3}{2}y + \frac{3}{2} = x + 2 \to x = \frac{3}{2}y - \frac{1}{2}$$

Plug into the budget constraint

$$130 - 4\left(\frac{3}{2}y - \frac{1}{2}\right) - 6y = 0 \to 130 - 6y + 2 - 6y = 0 \to 132 = 12y \to y^* = 11$$

Plugging back into the optimal condition (marginal rates of substitution set equal)
$$x^* = \frac{3}{2}(11) - \frac{1}{2} = 16$$

Problem 4.

First, note that $revenue = p_{output} * production\ function = 5xy^2$ and $cost = p_x x + p_y y = 10x + 6y$ and the constraint is that $x = y$ or $x - y = 0$.

a)
$$L = 5xy^2 - 10x - 6y + \lambda(x - y)$$

b) First Order Conditions:
$$\frac{\partial L}{\partial x} = 5y^2 - 10 + \lambda = 0 \to \lambda = 10 - 5y^2 \to$$

$$\frac{\partial L}{\partial y} = 10xy - 6 - \lambda = 0 \to \lambda = 10xy - 6 = 0$$

$$\frac{\partial L}{\partial \lambda} = x - y = 0$$

Now combine the first order conditions to obtain the optimality condition
$$10 - 5y^2 = 10xy - 6$$

Often, we want to solve for a price ratio and set this equal to some function of the variables, and then plug this into the constraint. In this case though, it is easier to plug the constraint into the optimality condition, so let's do that.
$$10 - 5x^2 = 10x^2 - 6 \to 16 = 15x^2 \to x^* = \sqrt{\frac{16}{15}} = y^*$$

Problem 5.

The Lagrangian function is

$$L = xyz + \lambda(45 - x - y - z) + \mu(y - 2x)$$

The first order conditions are:
$$L_\lambda : 45 - x - y - z = 0 \tag{1}$$

$$L_\mu : y - 2x = 0 \tag{2}$$

$$L_x : yz - \lambda - 2\mu = 0 \tag{3}$$

$$L_y : xz - \lambda + \mu = 0 \tag{4}$$

$$L_z : xy - \lambda = 0 \tag{5}$$

Rearranging (5) and plugging into (4), we get

$$xz - xy + \mu = 0 \rightarrow \mu = xy - xz \tag{6}$$

Plugging (5) and (6) and plugging into (3), we get

$$yz - xy - 2(xy - xz) = 0 \rightarrow yz = 3xy - 2xz$$

However, from (2), we know $y = 2x$, so

$$2xz = 6x^2 - 2xz \rightarrow 4xz = 6x^2 \rightarrow x = \frac{2}{3}z$$

Substituting into (1):

$$45 - x - 2x - \frac{3}{2}x = 0 \rightarrow 45 = \frac{9}{2}x \rightarrow x^* = 10$$

Then, $y^* = 2(10) = 20$ and $z^* = \frac{3}{2}(10) = 15$. So the critical point is $(x^*, y^*, z^*) = (10, 20, 15)$. So the stationary value is $f(x^*, y^*, z^*) = 10(20)(15) = 3000$.