Homework Day 3 Solutions - ECON 186

Problem 1. Chiang and Wainwright 5.1 #3, 5(a,b)

3)

a)

You could probably just look at the matrix and see that the rows are linearly independent, but to be sure, we want to reduce the matrices to row echelon form to know for sure.

$$\begin{bmatrix} 24 & 8 \\ 9 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/3 \\ 9 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$$

Since there are no nonzero rows, they are linearly independent.

b) Yes, the matrix is already in row echelon form, so we can see that there are 2 linearly independent rows.

c)

$$\begin{bmatrix} 0 & 4 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

There are 2 nonzero rows, so there are 2 linearly independent rows.

d)

$$\begin{bmatrix} -1 & 5 \\ 2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

Since there is a row of 0's in row echelon form, the rows are linearly dependent.

5)

a)

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ 0 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 3 \\ 0 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

There are no nonzero rows, so the rank of the matrix is 3 and it is nonsingular.

b)

$$\begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -4 \\ 6 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has a row of 0's after reducing to row echelon form, so the rank is 2 and therefore the matrix is singular.

Problem 2. Chiang and Wainwright 5.2 #1,2

1)

a) $\begin{vmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{vmatrix} = 8\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix} = -12 + 6 = -6$

b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 7 & 5 \\ 6 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 7 \\ 3 & 6 \end{vmatrix} = 1[7(9) - 6(5)] - 2[4(9) - 3(5)] +$ $3[4(6) - 3(7)] = 1[33] - 2[21] + 3[3] = 33 - 42 + 9 = 0$

c) $\begin{vmatrix} 4 & 0 & 2 \\ 6 & 0 & 3 \\ 8 & 2 & 3 \end{vmatrix} = 4\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 6 & 3 \\ 8 & 3 \end{vmatrix} + 2\begin{vmatrix} 6 & 0 \\ 8 & 2 \end{vmatrix} = 4[0(3) - 2(3)] + 2[12] = -24 + 24 = 0$

d) $\begin{vmatrix} 1 & 1 & 4 \\ 8 & 11 & -2 \\ 0 & 4 & 7 \end{vmatrix} = 1\begin{vmatrix} 11 & -2 \\ 4 & 7 \end{vmatrix} - 1\begin{vmatrix} 8 & -2 \\ 0 & 7 \end{vmatrix} + 4\begin{vmatrix} 8 & 11 \\ 0 & 4 \end{vmatrix} = 1[11(7) - 4(-2)] - 1[8(7) -$ $(-2)(0)] + 4[8(4) - 0(11)] = 1[85] - 1[56] + 4[32] = 85 - 56 + 128 = 157$

e) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix} = a(cb - a^2) - b(b^2 - ac) + c(ba - c^2) =$ $3abc - a^3 - b^3 - c^3$

f) $\begin{vmatrix} x & 5 & 0 \\ 3 & y & 2 \\ 9 & -1 & 8 \end{vmatrix} = x\begin{vmatrix} y & 2 \\ -1 & 8 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 9 & 8 \end{vmatrix} + 0\begin{vmatrix} 3 & y \\ 9 & -1 \end{vmatrix} = x[8y+2] - 5[3(8) - 9(2)] =$
$8xy + 2x - 30$

2)

$+, -, +, -, -$

Problem 4. Chiang and Wainwright 5.4 #4a (solve using both the cofactor method and the augmented matrix method)

First, let's find the inverse using the cofactor method.
We want to find the inverse of the matrix

$$\begin{vmatrix} 4 & -2 & 1 \\ 7 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

Step 1: Construct a matrix of minors.

$$\begin{bmatrix} 3(1)-0(0) & 7(1)-2(0) & 7(0)-3(2) \\ -2(1)-1(0) & 4(1)-1(2) & 4(0)+2(2) \\ -2(0)-1(3) & 4(0)-1(7) & 4(3)+2(7) \end{bmatrix} = \begin{bmatrix} 3 & 7 & -6 \\ -2 & 2 & 4 \\ -3 & -7 & 26 \end{bmatrix}$$

Step 2: Find the matrix of cofactors.

$$\begin{bmatrix} 3 & -7 & -6 \\ 2 & 2 & -4 \\ -3 & 7 & 26 \end{bmatrix}$$

Step 3: Transpose the matrix of cofactors.

$$\begin{bmatrix} 3 & 2 & -3 \\ -7 & 2 & 7 \\ -6 & -4 & 26 \end{bmatrix}$$

Step 4: Find the determinant of the original matrix. Since we already know the matrix of minors, we just have to multiply the top row by their corresponding minors.

$$4(3) + 2(7) + 1(-6) = 12 + 14 - 6 = 20$$

Then applying the formula

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

we get

$$A^{-1} = \frac{1}{20}\begin{bmatrix} 3 & 2 & -3 \\ -7 & 2 & 7 \\ -6 & -4 & 26 \end{bmatrix}$$

Augmented matrix method:

$$\left[\begin{array}{ccc|ccc} 4 & -2 & 1 & 1 & 0 & 0 \\ 7 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & 1/4 & 0 & 0 \\ 7 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 13/2 & -7/4 & -7/4 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 & 1 \\ 0 & 13/2 & -7/4 & -7/4 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & -5 & 3/2 & 1 & -13/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & -3/10 & -1/5 & 13/10 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/20 & 1/10 & -3/20 \\ 0 & 1 & 0 & -7/20 & 1/10 & 7/20 \\ 0 & 0 & 1 & -3/10 & -1/5 & 13/10 \end{array}\right]$$

It is easy to check that we got the same inverse matrix using both methods.

Problem 5. Chiang and Wainwright 5.5 #3a

3)

a)
In this case,

$$A = \begin{bmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad d = \begin{bmatrix} 16 \\ 5 \\ 7 \end{bmatrix}$$

Cramer's Rule says that

$$x_j^* = \frac{|A_j|}{|A|}$$

where $|A_j|$ is the jth column replaced by d.
So,

$$|A| = \begin{vmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix} = 8\begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix}$$

$$= 8(6) - 10 = 48 - 10 = 38$$

$$|A_1| = \begin{vmatrix} 16 & -1 & 0 \\ 5 & 2 & 5 \\ 7 & 0 & 3 \end{vmatrix} = 16\begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 5 & 5 \\ 7 & 3 \end{vmatrix}$$

$$= 16(6) + 15 - 35 = 96 + 15 - 35 = 76$$

$$|A_2| = \begin{vmatrix} 8 & 16 & 0 \\ 0 & 5 & 5 \\ 2 & 7 & 3 \end{vmatrix} = 8\begin{vmatrix} 5 & 5 \\ 7 & 3 \end{vmatrix} - 16\begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix}$$

$$= 8(15 - 35) - 16(-10) = -160 + 160 = 0$$

$$|A_3| = \begin{vmatrix} 8 & -1 & 16 \\ 0 & 2 & 5 \\ 2 & 0 & 7 \end{vmatrix} = 8\begin{vmatrix} 2 & 5 \\ 0 & 7 \end{vmatrix} + \begin{vmatrix} 0 & 5 \\ 2 & 7 \end{vmatrix} + 16\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= 8(14) - 10 + 16(-4) = 112 - 10 - 64 = 38$$

$$x_1^* = 2,\ x_2^* = 0,\ x_3^* = 1$$