ECON 186 Class Notes: Linear Algebra

Jijian Fan

Matrix Algebra

- Two matrices are equal if and only if they have the same dimension and identical elements in corresponding locations.
- Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$
- Two matrices can be added or subtracted if and only if they have the same dimension. Then addition or subtraction is performed by adding or subtracting each corresponding element.
- Examples:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix} - \begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 6 \end{bmatrix}$$

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \end{bmatrix}$ cannot be added.

Matrix Algebra

- Scalar multiplication is to multiply each element of the matrix by a scalar.
- Example 1:

$$b\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ba_{11} & ba_{12} \\ ba_{21} & ba_{22} \end{bmatrix}$$

- Example 2:

$$7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}$$

Matrix Algebra

- Suppose we want to multiply two matrices, A and B to form product AB. They are conformable for multiplication (we are able to multiply them) if and only if the column dimension of A (the lead matrix) is equal to the row dimension of B (the lag matrix).
- Example: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$
  - The product AB is defined since A has dimension 2×2 and thus has 2 columns and B has dimension 2×3 and thus has 2 rows.
  - The product BA is however not defined since B has 3 columns while A has only 2 rows.
- A product will have the same number of rows as the lead matrix and columns as the lag matrix.
  - Let A have dimension $m \times n$ and B have dimension $n \times q$ then AB would have dimension $m \times q$.
  - In the above example, AB would have dimension 2×3.

Matrix Algebra

- Each element in a product is defined as a sum of a products, in which the elements across the rows of A are multiplied with the elements down the column of B.
- Example 1: Again, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

- Example 2: Let $A = \begin{bmatrix} 4 & 7 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 6 & -2 \end{bmatrix}$

$$AB = \begin{bmatrix} 4(1)+7(2) & 4(3)+7(6) & 4(-3)+7(-2) \\ (-1)(1)+2(2) & (-1)(3)+2(6) & (-1)(-3)+2(-2) \end{bmatrix} =$$

$$= \begin{bmatrix} 18 & 54 & -26 \\ 3 & 9 & -1 \end{bmatrix}$$

Matrix Algebra

- Example 3: Let $A = \begin{bmatrix} 7 & 10 & -1 \\ 2 & 1 & 3 \\ 4 & -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix}$
- Find product BA. Is it conformable for multiplication?
    - Yes, it is conformable for multiplication since the lead matrix B has 3 columns and the lag matrix A has 3 rows. The product BA will have dimension 2×3.
    - $BA =$
      $$\begin{bmatrix} 0*7+1*2+2*4 & 0*10+1*1+2*-3 & 0*-1+1*3+2*5 \\ -1*7+-2*2+0*4 & -1*10+-2*1+0*-3 & -1*-1+-2*3+0*5 \end{bmatrix}$$
      $$= \begin{bmatrix} 10 & -5 & 13 \\ -11 & -12 & -5 \end{bmatrix}$$

Matrix Algebra

- Recall our previous example where

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

- Now that we understand matrix multiplication, we can see why we can write the original system of equations as $Ax = d$.
- $Ax = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 - 2x_3 \\ 4x_1 - x_2 + 5x_3 \end{bmatrix} = d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$

A few notes about vectors and division

- Vectors are simply special cases of matrices, and thus are multiplied with the same rules.
- Example: Let $u = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$ and $u' = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Note that u has dimension 1×3 while u' has dimension 3×1. They are conformable since $3 = 3$ and the dimension of uu' will be 1×1, that is, a scalar. Find uu'.

$$uu' = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3(3) + 6(6) + 9(9) = 126$$

- Matrices cannot be divided because A/B would actually represent both AB^{-1} or $B^{-1}A$, where B^{-1} is the inverse of B (which we will discuss later). While one of these could be defined for the given matrices, the other may not be, so the division of matrices is not defined.

Linear Dependence

- A set of vectors $v_1, ..., v_n$ is linearly dependent if and only if any one of them can be expressed as a linear combination of the remaining vectors. Otherwise they are linearly independent.
- Example 1: Let $v_1 = \begin{bmatrix} 5 & 12 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 10 & 24 \end{bmatrix}$.

$$2v_1 = 2\begin{bmatrix} 5 & 12 \end{bmatrix} = \begin{bmatrix} 10 & 24 \end{bmatrix} = v_2$$

Linear Dependence

- Example 2: Let $v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. We may write:
$$3v_1 - 2v_2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v_3$$
- Thus, v_3 is a linear combination of v_1 and v_2, so these vectors are linearly dependent.
- Application: Perfect multicollinearity in econometrics. If we think of each column of data in a dataset as a column vector, then if two variables are so similar that they are simply linear combinations of each other, then they are perfectly multicollinear and we cannot proceed with our regression.

Commutative, Associative, and Distributive Laws

- Matrix addition is commutative. $A+B=B+A$.
  - Example: Let $A=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$. Then $A+B=\begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}=B+A$.
- Matrix addition is associative. $(A+B)+C=A+(B+C)$
  - Example: Let $v_1=\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2=\begin{bmatrix} 9 \\ 1 \end{bmatrix}$, $v_3=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

    $$(v_1+v_2)-v_3=\begin{bmatrix} 12 \\ 5 \end{bmatrix}-\begin{bmatrix} 2 \\ 5 \end{bmatrix}=\begin{bmatrix} 10 \\ 0 \end{bmatrix}=v_1+(v_2-v_3)=\begin{bmatrix} 3 \\ 4 \end{bmatrix}+\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Commutative, Associative, and Distributive Laws

- Matrix multiplication is not commutative. $AB \neq BA$.
- Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1(0)+2(6) & 1(-1)+(2)(7) \\ 3(0)+4(6) & 3(-1)+4(7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0(1)+(-1)(3) & 0(2)+(-1)(4) \\ 6(1)+7(3) & 6(2)+7(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Commutative, Associative, and Distributive Laws

- Matrix multiplication is associative. $(AB)C = A(BC) = ABC$. In order to multiply three or more matrices, each adjacent pair of matrices must be conformable.
- If the conformability condition is met, adjacent pair of matrices may be multiplied in any order.
- Example: Let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ 5 & 2 \\ 7 & 1 \end{bmatrix}$ and suppose we want to form ABC. Since A has dimension 2×1 and C has dimension 3×2, B must have dimension 1×3. Then, let $B = \begin{bmatrix} 3 & 2 & -2 \end{bmatrix}$.

Commutative, Associative, and Distributive Laws

- $(AB)C = \begin{bmatrix} 1(3) & 1(2) & 1(-2) \\ 2(3) & 2(2) & 2(-2) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 5 & 2 \\ 7 & 1 \end{bmatrix} =$

  $\begin{bmatrix} 3 & 2 & -2 \\ 6 & 4 & -4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 5 & 2 \\ 7 & 1 \end{bmatrix} =$

  $\begin{bmatrix} 3(0)+2(5)+(-2)(7) & 3(-1)+2(2)+(-2)(1) \\ 6(0)+4(5)+(-4)(7) & 6(-1)+4(2)+(-4)(1) \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix}$

- $A(BC) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3(0)+2(5)+(-2)(7) & 3(-1)+2(2)+(-2)(1) \end{bmatrix} =$

  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -4 & -1 \end{bmatrix} = \begin{bmatrix} 1(-4) & 1(-1) \\ 2(-4) & 2(-1) \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix}$

- Matrix multiplication is distributive. $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$. The conformability conditions for addition and multiplication must be met in each step.

Special Types of Matrices

- A square matrix is a matrix which has the same number of rows and columns. That is, a matrix of dimension $n \times n$.
- The identity matrix is a square matrix with 1's in its principal diagonal (the diagonal running from the top left to the bottom right of the matrix) and 0s everywhere else. This matrix is denoted I.
  - Example: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
  - The identity matrix plays a very similar role to the number 1 in scalar algebra. That is, for any matrix A, $IA = AI = A$. Alternatively, using the associative law, $AIB = (AI)B = AB$.

Special Types of Matrices

- Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+2(0) & 1(0)+2(1) \\ 3(1)+4(0) & 3(0)+4(1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(1)+0(3) & 1(2)+0(4) \\ 0(1)+1(3) & 0(2)+1(4) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Special Types of Matrices

- The null matrix is any matrix with 0 as every element. It plays the role of the number 0 in scalar algebra.
  - $\underset{(2x2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\underset{(3x2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
- Example 1: $A + 0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$
- Example 2:

$$\underset{(2x3)}{A} \; \underset{(3x1)}{0} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underset{(2x1)}{0}$$

Special Types of Matrices

- An upper triangular matrix is a square matrix where all the entries below the principal diagonal are 0.
  - Example: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$
- A lower triangular matrix is a square matrix where all the entries above the principal diagonal are 0.
  - Example: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$
- A triangular matrix is a matrix that is either upper or lower triangular.
- A matrix can be upper and lower triangular only if the only non-zero elements are on the principal diagonal. The identity matrix is an important example.
- Application: Triangular matrices are much easier to solve numerically using a process called forward and back substitution, and does not require inversion as for non-triangular matrices.

Tranposes of Matrices

- The transpose of a matrix A of dimension $m \times n$ is a matrix $A^{'}$ of dimension $n \times m$ obtained from A by interchanging the rows with the columns.
  - Example: Let $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$.
  - $A^{'} = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$ and $B' = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$
- Properties of Transposes

$$(A')' = A$$

$$(A+B)' = A' + B'$$

$$(AB)' = B'A'$$

Example of Properties of Transposes

- Let $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} \rightarrow (A')' = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$$

$$(A+B)' = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}' = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix} = A' + B' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix}$$

$$(AB)' = \left(\begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix} \right)' = \begin{bmatrix} 4(2)+1(7) & 4(0)+1(1) \\ 9(2)+0(7) & 9(0)+0(1) \end{bmatrix}' =$$
$$\begin{bmatrix} 15 & 1 \\ 18 & 0 \end{bmatrix}' = \begin{bmatrix} 15 & 18 \\ 1 & 0 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2(4)+7(1) & 2(9)+7(0) \\ 0(4)+1(1) & 0(9)+1(0) \end{bmatrix} = \begin{bmatrix} 15 & 18 \\ 1 & 0 \end{bmatrix}$$

Matrix Inverses

- A square matrix A has an inverse A^{-1} if the following condition is met: $AA^{-1} = A^{-1}A = I$.
- Properties of inverses:
  - Squareness is a necessary but not sufficient condition for the existence of an inverse. If a square matrix A has an inverse, A is nonsingular. If A does not have an inverse, it is singular.
  - If A^{-1} does exist, then A and A^{-1} are inverses of each other.
  - A and A^{-1} will always have the same dimension.
  - If an inverse exists, then it is unique.
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A')^{-1} = (A^{-1})'$
- Proofs are on pages 75-77.

Matrix Inverses

- Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 3(2)+1(0) & 3(-1)+1(3) \\ 0(2)+2(0) & 0(-1)+2(3) \end{bmatrix}\frac{1}{6} =$$
$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2(3)+(-1)(0) & 2(1)+(-1)(2) \\ 0(3)+3(0) & 0(1)+3(2) \end{bmatrix} =$$
$$\frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Inverses

- As we discussed previously, we are able to write systems of equations as $Ax = d$ as long as the system meets the proper conditions.
- Then, if A^{-1} exists, we can premultiply both sides by A^{-1} to get
  - $A^{-1}Ax = A^{-1}d \rightarrow x = A^{-1}d$
  - The left hand side is now a column vector of variables and the right hand side is a matrix of solution values for the system of equations.
  - We now see that if we are able to find the inverse of the coefficient matrix for a system of equations, we are able to determine the solutions for that system.
- We will return shortly to how to determine the existence of inverses and calculate them.